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Fourier-Jacobi expansion and Ikeda lifting

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1 Introduction and Main results

We consider the following map $\Psi^{(2n-1)}$ from elliptic modular forms to Siegel modular forms of half-integral weight, which is the decomposition of the following three maps :

$$\Psi^{(2n-1)} \;:\; S_{2k}(SL_2(\mathbb{Z})) \to S_{k+n}(\Gamma_{2n}) \to J^{cusp}_{k+n,1}(\Gamma^J_{2n-1}) \to S^+_{k+n-\frac{1}{2}}(\Gamma_0^{(2n-1)}(4)) \,.$$

(For the notations, see below.) To study this map was suggested to the author by Professor T. Ikeda.

The purpose of this article is to show the following two results :

1. The map $\Psi^{(2n-1)}$ maps normalized Hecke eigenforms to Hecke eigenforms. Moreover, the L-function of a Hecke eigenform and its image under $\Psi^{(2n-1)}$ are related by an explicit formula. (cf. Theorem 1.)

2. The Fourier-Jacobi coefficients of the image under the Ikeda lifting can be written explicitly in terms of the *first* Fourier-Jacobi coefficient, by using certain Hecke operators which increase the index of Jacobi forms. (cf. Theorem 2.)

We remark that the second statement was already known to Yamazaki [10] in the case of Siegel-Eisenstein series. In fact, we use his theorem to show the second statement.

We explain our results more precisely. Let $k+n$ $(k, n \in \mathbb{N})$ be an even integer and let $f \in S_{2k}(SL(2,\mathbb{Z}))$ be a normalized Hecke eigenform of weight $2k$. We denote by $I(f) \in S_{k+n}(Sp(2n,\mathbb{Z}))$ the image of f under the Ikeda lifting.

We put $e(*) := \exp(2\pi i *)$, and we denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n. We denote by ϕ_r the r-th Fourier-Jacobi coefficient of $I(f)$, namely,

$$I(f)\left(\left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right)\right) \;=\; \sum_{r>0} \phi_r(\tau, z) e(r\tau') \quad \left(\left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right) \in \mathfrak{H}_{2n} \,,\ \tau \in \mathfrak{H}_{2n-1} \,,\ \tau' \in \mathfrak{H}_1\right),$$

where $\phi_r \in J^{cusp}_{k+n,r}(\Gamma^J_{2n-1})$ is a Jacobi cusp form of weight $k+n$ of index r of degree $2n-1$. Associated for f we have the Siegel modular form (in the plus space) $\Psi^{(2n-1)}(f) \in S^+_{k+n-1/2}(\Gamma_0^{(2n-1)}(4))$ of weight $k+n-1/2$ of degree $2n-1$, which corresponds to the first Fourier-Jacobi coefficient $\phi_1 \in J^{cusp}_{k+n,1}(\Gamma^J_{2n-1})$ of $I(f)$ by the isomorphism between the space of Jacobi forms of index 1 and the plus space. (cf. [3], [6], [9], see also subsection 2.3.)

We have the following two Theorems.

Theorem 1. *Let $f \in S(SL_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then the form $\Psi^{(2n-1)}(f)$ is a Hecke eigenform, and its L-function satisfies the following identity up to the Euler 2-factors :*

$$L(s, \Psi^{(2n-1)}(f)) = \prod_{i=0}^{2n-2} L(s-i, f).$$

Here $L(s,f)$ is the usual L-function of f, and the L-function of $\Psi^{(2n-1)}(f)$ is the one introduced by Zhuravlev [12], [13] (and will be recalled in subsection 2.2.)

We denote by α_p the Satake parameter of f, which is determined by the identity $\alpha_p + \alpha_p^{-1} = a_f(p)p^{-k+1/2}$, where $a_f(p)$ is the p-th Fourier coefficient of f. We obtain the following Theorem.

Theorem 2. *Let $f \in S(SL_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then for any positive integer r, the r-th Fourier-Jacobi coefficient ϕ_r of $I(f)$ satisfies the identity :*

$$\phi_r = \phi_1|_{k+n} D_{2n-1}(r, \{\alpha_p\}_p) ,$$

where the $D_{2n-1}(r, \{\alpha_p\}_p)$ are defined by

$$\sum_{r>0} \frac{D_{2n-1}(r,\{\alpha_p\})}{r^s} =$$
$$\prod_{p:prime} (1 - G_p(\alpha_p)T(p)p^{\frac{1}{2}(n-1)(n+2)-s} + T_{0,2n-1}(p^2)p^{2n(2n-1)-1-2s})^{-1} .$$

Here $T(p)$ and $T_{0,2n-1}(p^2)$ are Hecke operators (introduced by Yamazaki [10], [11], and whose precise definition will be recalled in subsection 2.3), and for each p, we use

$$G_p(\alpha_p) = \prod_{i=1}^{n-1} \{(1+\alpha_p p^{\frac{1}{2}-i})(1+\alpha_p^{-1}p^{\frac{1}{2}-i})\}^{-1} ,$$

for $n > 1$, $G_p(\alpha_p) = 1$ for $n = 1$.

We remark that the above Theorem gives a generalization of Yamazaki's theorem (see subsection 2.3) on Siegel cusp forms obtained from elliptic modular forms by Ikeda lifting.

The main tool of the proof of the above theorems is the study of the Fourier coefficients of $I(f)$ using Eisenstein series.

2 Notations and proofs

2.1 Ikeda lifting

The existence of the Ikeda lifting was first conjectured by Duke-Imamoglu and was shown by Ikeda [7]. Following [7], we shall introduce some notations. Let f be a cusp form of weight $2k$ with respect to $SL(2,\mathbb{Z})$, assume that f is a normalized Hecke eigenform. We fix a positive integer n which satisfies $k+n \in 2\mathbb{Z}$. For a positive-definite half-integral symmetric matrix B, we put

$$A(B) := c(\delta_B) f_B^{k-1/2} \prod_p \tilde{F}_p(B, \alpha_p),$$

where $c(\delta_B)$, f_B are certain constants and $\tilde{F}_p(B, X_p)$ is a certain Laurent-polynomial of X_p which corresponds to Siegel series, and where α_p is the Satake parameter of f. More precisely, δ_B is the absolute value of the discriminant of the quadratic field $\mathbb{Q}(\sqrt{(-1)^n \det(2B)})$, and f_B is the positive integer which is determined by the identity $\det(2B) = \delta_B f_B^2$, and $c(\delta_B)$ is the δ_B-th Fourier-coefficient of the modular form of half-integral weight which corresponds to f under the Shimura correspondence. It is known that the Laurent-polynomial $\tilde{F}_p(B, X_p)$ satisfies the functional equation $\tilde{F}_p(B, X_p) = \tilde{F}_p(B, X_p^{-1})$ for any B.

The following Theorem is known.

Theorem 3 (Ikeda [7]). *The form* $(I(f))(\tau) := \sum_B A(B) e(B\tau)$ $(\tau \in \mathfrak{H}_{2n})$ *is a Siegel modular form of weight* $k+n$ *of degree* $2n$. *Moreover* $I(f)$ *is a Hecke eigenform, and its standard L-function satisfies* $L(s, I(f)) = \prod_{i=1}^{2n} L(s+k+n-i, f)$.

2.2 Jacobi forms of higher degree and Siegel modular forms of half-integral weight

We need some notations to describe the definitions of Jacobi forms and the plus space. Let $G_n^J \subset Sp(n+1,\mathbb{R})$ be the Jacobi group defined by

$$G_n^J := \{M \in Sp(n+1,\mathbb{R}) \mid \text{The last row of } M \text{ is } (0,...,0,1)\} .$$

We set $\Gamma_n^J := G_n^J \cap Sp(n+1,\mathbb{Z})$.

Let $\phi(\tau, z)$ be a holomorphic function on $\mathfrak{H}_n \times \mathbb{C}^n$, where we regard z as a column vector. By definition, we call the form ϕ a Jacobi cusp form of weight k of index m of degree n, if the form $\tilde{\phi}\left(\left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right)\right) := \phi(\tau, z) e(m\tau')$ $\left(\left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right) \in \mathfrak{H}_{n+1}\right)$ satisfies the identity $\tilde{\phi}|_k \gamma = \tilde{\phi}$ for any $\gamma \in \Gamma_n^J$ and satisfies the well-known cusp condition. (In the case of $n > 1$ the cusp condition is automatically fulfilled by the Koecher-Principle. (cf. Ziegler [14].) We denote by $J_{k,m}^{cusp}(\Gamma_n^J)$ the space of Jacobi forms of weight k, of index m and of degree n.

The plus space is a certain subspace of Siegel modular forms of half-integral weight introduced by Kohnen [8] in the case of degree 1, and generalized for higher degree by Ibukiyama [6]. We denote by $S_{k-1/2}(\Gamma_0^{(n)}(4))$ the space of cusp forms of Siegel modular forms of weight $k-1/2$ of degree n with level 4. We denote the plus space of weight $k-1/2$ of degree n by $S^+_{k-1/2}(\Gamma_0^{(n)}(4))$, which is the subspace of $S_{k-1/2}(\Gamma_0^{(n)}(4))$ defined by

$$S^+_{k-1/2}(\Gamma_0^{(n)}(4)) = \left\{ F \in S_{k-1/2}(\Gamma_0^{(n)}(4)) \mid \begin{array}{c} A(F,N)=0 \text{ unless} \\ \exists \lambda \in \mathbb{Z}^n \text{ s.t } N + \lambda^t\lambda \in 4\,\mathrm{Sym}_n^* \end{array} \right\},$$

where $A(F,N)$ is the N-the Fourier coefficient of F, and where Sym_n^* denotes the set of all half-integral symmetric matrices of size n.

It is known that the space of Jacobi cusp forms of index 1 of weight k of degree n is linearly isomorphic to the plus space of degree n of weight $k-1/2$. (cf. Eichler-Zagier [3] for $n=1$, Ibukiyama [6] for $n>1$, and also Takase [9] by using representation theory.) This isomorphism is Hecke-equivalent. By virtue of this isomorphism, the Fourier coefficients of Jacobi forms of index 1 coincide with those of Siegel modular forms of half-integral weight.

Let $G \in S_{k-1/2}(\Gamma_0^{(n)}(4))$ be a Hecke eigenform. We define $L(s,G)$ by

$$L(s,G) := \prod_{p\neq 2}\prod_{i=1}^{n} \left\{(1-\alpha_{i,p}p^{-s+k-3/2})(1-\alpha_{i,p}^{-1}p^{-s+k-3/2})\right\}^{-1},$$

where $\alpha_{i,p}^{\pm}$ are the Satake parameters of G (cf. Zhuravlev [12], [13].)

2.3 Hecke operators acting on the space of Jacobi forms and Yamazaki's theorem

We define $GSp^+(n,\mathbb{R})$ by :

$$GSp^+(n,\mathbb{R}) := \left\{M \in GL(2n,\mathbb{R}) \mid MJ_n^tM = \nu J_n \text{ for some } \nu > 0\right\},$$

where $J_n = \left(\begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix}\right)$, and we write $\nu(M)=\nu$. For a holomorphic function F on $\mathfrak{H}_n$ and for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GSp^+(n,\mathbb{R})$, we define the operator $|_k$ by : $(F|_kM)(\tau) := \det(M)^{\frac{k}{2}}\det(C\tau+D)^{-k}F((A\tau+B)(C\tau+D)^{-1})$.

We let $\rho : GSp^+(n,\mathbb{R}) \to GSp^+(n+1,\mathbb{R})$ by $\rho(M) := \begin{pmatrix} A & & B & \\ & \nu(M) & & \\ C & & D & \\ & & & 1 \end{pmatrix}$, where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GSp^+(n,\mathbb{R})$.

Let $\phi \in J_{k,m}^{cusp}(\Gamma_n^J)$ be a Jacobi form of weight k of index m of degree n. For $M \in GSp^+(n,\mathbb{Q}) \cap M(2n,\mathbb{Z})$, we define the action of the double coset $\Gamma_n^J\rho(M)\Gamma_n^J$ by $\phi|\Gamma_n^J\rho(M)\Gamma_n^J := \sum_i \phi|_kM_i$, where $\Gamma_n^J\rho(M)\Gamma_n^J = \bigcup_i \Gamma_n^J M_i$ is the right Γ_n^J-coset decomposition of $\Gamma_n^J\rho(M)\Gamma_n^J$.

Following Ibukiyama [6] and Yamazaki [10], we define three operators $T_s(p^2)$ $(s = 0, ..., n)$ (cf. [6]), $T(p)$ and $T_{0,n}(p^2)$ (cf. [10]) as follows :

$$\begin{aligned}
\phi|T_s(p^2) &:= p^{kn} \sum_{\lambda,\mu\in(\mathbb{Z}p/\mathbb{Z})^n} e({}^t\lambda\tau\lambda + 2{}^tz\lambda) \sum_{\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right)\in Sp(n,\mathbb{Z})\backslash Sp(n,\mathbb{Z})k_{p,s}Sp(n,\mathbb{Z})} \\
&\quad \times \det(D)^{-k}\phi((A\tau+B)D^{-1}, p{}^tD^{-1}(z+\tau\lambda+\mu)), \\
\phi|T(p) &:= p^{-n(n+1)/2} \sum_{\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right)\in Sp(n,\mathbb{Z})\backslash Sp(n,\mathbb{Z})M_pSp(n,\mathbb{Z})} ((\phi(\tau,z)e(m\tau'))|_k\rho\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right))\, e(-mp\tau'), \\
\phi|T_{0,n}(p^2) &:= p^{-n(n+1)}\left((\phi(\tau,z)e(m\tau'))|_k\begin{pmatrix} p1_n & & & \\ & p^2 & & \\ & & p1_n & \\ & & & 1\end{pmatrix}\right) e(-mp^2\tau'),
\end{aligned}$$

where $k_{p,s} = \begin{pmatrix} 1_{n-s} & & & \\ & p1_s & & \\ & & p^21_{n-s} & \\ & & & 1_s\end{pmatrix}$, and where $M_p := \begin{pmatrix} 1_n & \\ & p1_n\end{pmatrix}$.

Then we have the following Lemma.

Lemma 1. *For each $\phi \in J^{cusp}_{k,m}(\Gamma^J_n)$ and for each p the following identity hold :*

$$\begin{aligned}
\phi|T_s(p^2) &= c_1(p)\,\phi|\Gamma^J_n\widetilde{k_{p,s}}\Gamma^J_n, \\
\phi|T(p) &= c_2(p)\,\phi|\Gamma^J_n\begin{pmatrix} 1_n & & & \\ & p & & \\ & & p1_n & \\ & & & 1\end{pmatrix}\Gamma^J_n, \\
\phi|T_{0,n}(p^2) &= c_3(p)\,\phi|\Gamma^J_n\begin{pmatrix} p1_n & & & \\ & p^2 & & \\ & & p1_n & \\ & & & 1\end{pmatrix}\Gamma^J_n,
\end{aligned}$$

where $\widetilde{k_{p,s}} = diag(1_{n-s}, p1_s, p, p^21_{n-s}, p1_s, p)$. Here the $c_j(p)$ are constants (not depending on ϕ.)

Proof. This follows from a direct calculation of representatives of left Γ^J_n-coset of the double-cosets of the right hand side. □

We call a Jacobi form ϕ a Hecke eigenform if ϕ is an eigenform for any $T_s(p^2)$.

The above operators also act on the space of non-cusp forms. As for Siegel-Eisenstein series, the following Theorem is known.

Theorem 4 (Yamazaki [10]). *Let $k > 2n+1$ be an even integer and for $r > 0$ let $e^{(2n-1)}_{k,r}$ be the r-th Fourier-Jacobi coefficient of Siegel Eisenstein series $E^{(2n)}_k$ of weight k of degree 2n (i.e. $E^{(2n)}_k(\left(\begin{smallmatrix} \tau & z \\ {}^tz & \tau'\end{smallmatrix}\right)) = \sum_{r\geq 0} e^{(2n-1)}_{k,r}(\tau,z)e(r\tau')$.) Then we have the following identity :*

$$e^{(2n-1)}_{k,r} = e^{(2n-1)}_{k,1}|_kD_{2n-1}(r,\{p^{k-n-\frac{1}{2}}\}_p)\ .$$

(Here the $D_{2n-1}(r,\{p^{k-n-\frac{1}{2}}\}_p)$ are the operators introduced in Theorem 2.)

We remark that a similar identity was also shown for odd integers instead of $2n$. However in this article we treat only the case of even integers.

2.4 The proof of Theorem 1

We prove Theorem 1. Let $k > n+2$ be an even integer and let $E_{k,r}^{(n)}$ be the Jacobi-Eisenstein series of weight k of index r of degree n. This Jacobi-Eisenstein series was first introduced by Eichler-Zagier [3] in the case $n = 1$ and was generalized for higher degree by Ziegler [14]. Let $e_{k,1}^{(n)}$ be the first Fourier-Jacobi coefficient of Siegel-Eisenstein series of even weight k of degree $n+1$. By Satz 7 of Boecherer [1] (cf. also Yamazaki [10] Theorem 5.5), we have that the first Fourier-Jacobi coefficient $e_{k,1}^{(n)}$ coincides with the Jacobi-Eisenstein series $E_{k,1}^{(n)}$ of index 1.

Moreover, by using Lemma 1 and by using an argument as in Freitag [4](Bemerkung 4.7 p.268), we have that the Siegel-Eisenstein series $E_{k,1}^{(n)}$ is an eigenform for any operator $T_s(p^2)$. Therefore we conclude that the first Fourier-Jacobi coefficient $e_{k,1}^{(n)}$ is also a Hecke eigenform.

The main idea of the proof of Theorem 1 is to deduce certain properties of $\tilde{F}_p(B, X_p)$ from properties of Siegel-Eisenstein series. The following lemma was shown by Ikeda [7], and play an important rule to the proofs of Theorem 1 and Theorem 2.

Lemma 2. *Let $F(\{X_p\}) \in \mathbb{C}[X_2 + X_2^{-1}, X_3 + X_3^{-1}, X_5 + X_5^{-1}, \ldots]$ be a Laurent-Polynomial. If F satisfies $F(\{p^{k-1/2}\}) = 0$ for sufficiently many integers k, then $F(\{X_p\}) = 0$.*

Proof. It is not difficult to show this and the details will be omitted here. □

Let B be a positive-definite half-integral symmetric matrix of size $2n$, then it is known that the B-th Fourier coefficient $A(E_k^{(2n)}, B)$ of the Siegel-Eisenstein series $E_k^{(2n)}$ of weight k of degree $2n$ can be written as follows :

$$A(E_k^{(2n)}, B) = h_{k-n-1/2}(\delta_B) f_B^{k-n-1/2} \prod_{p | f_B} \tilde{F}_p(B, p^{k-n-1/2}).$$

Here $h_{k-n-1/2}(\delta_B)$ is the δ_B-th Fourier coefficient of the Cohen-Eisenstein series of weight $k-n-1/2$ (cf. Cohen [2].)

For a positive integer m, we define two sets by

$$\begin{aligned} S_{n,m} &:= \left\{(N,R) \in \mathrm{Sym}_n^* \times \mathbb{Z}^n \,|\, N \geq 0\,,\, 4Nm - R^t R \geq 0\right\}, \\ S_{n,m}^+ &:= \left\{(N,R) \in S_{n,m} \,|\, 4Nm - R^t R > 0\right\}. \end{aligned}$$

Let $\phi \in J_{k,m}(\Gamma_n^J)$ be a Jacobi form and let $(N,R) \in S_{n,m}$. We denote by $A(\phi, (N,R))$ the (N,R)-th Fourier coefficient of ϕ, that is,

$$\phi(\tau, z) = \sum_{(N,R) \in S_{n,m}} A(\phi, (N,R)) e(N\tau + R^t z) \quad ((\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^n.)$$

Let $(N,R)\in S^{+}_{2n-1,1}$, and put $B_1=\left(\begin{smallmatrix} N & R\\ {}^tR & 1\end{smallmatrix}\right)$. The (N,R)-th Fourier-coefficient of $e^{(2n-1)}_{k,1}$ can be written as

$$(2.1)\qquad A(e^{(2n-1)}_{k,1},(N,R))=h_{k-n-1/2}(\delta_{B_1})f_{B_1}^{k-n-1/2}\prod_{p|f_{B_1}}\tilde{F}_p(B_1,p^{k-n-1/2}).$$

Let $\Gamma^J_n\widetilde{k_{q,s}}\Gamma^J_n$ be a double-coset as defined in subsection 2.3. For a Jacobi form $\phi\in J_{k,m}(\Gamma^J_n)$, we denote by $A(\phi,(N,R),\widetilde{k_{q,s}})$ the (N,R)-th Fourier coefficient of $\phi|\Gamma^J_n\widetilde{k_{q,s}}\Gamma^J_n$.

By a direct calculation, we find that the (N,R)-th Fourier-coefficient of the form $e^{(2n-1)}_{k,1}|\Gamma^J_n k_{s,q}\Gamma^J_n$ can be written as the form :

$$(2.2)\qquad \begin{aligned} &A(e^{(2n-1)}_{k,1},(N,R),k_{s,q})\\ &=h_{k-n-1/2}(\delta_{B_1})f_{B_1}^{k-n-1/2}\sum_i\beta_i\prod_{p|f_{B_{i,1}}}\tilde{F}_p(B_{i,1},p^{k-n-1/2}),\end{aligned}$$

where β_i are certain constants, and where $B_{i,1}$ are certain matrices of the form $B_{i,1}=\left(\begin{smallmatrix} N_i & R_i\\ {}^tR_i & 1\end{smallmatrix}\right)\in M_{2n}(\mathbb{Z})$. These β_i and $B_{i,1}$ depend only on the choice of (N,R) and of $\Gamma^J_n\widetilde{k_{q,s}}\Gamma^J_n$. Because $e^{(2n-1)}_{k,1}$ is a Hecke eigenform for any even integer $k>2n+1$, (using Lemma 2, and identities (2.1), (2.2)) we have that there exists a certain Laurent polynomial $\Phi(\Gamma^J_n\widetilde{k_{q,s}}\Gamma^J_n,X_q)$ which satisfies :

$$(2.3)\qquad \Phi(\Gamma^J_n\widetilde{k_{q,s}}\Gamma^J_n,X_q)\prod_{p|f_{B_1}}\tilde{F}_p(B_1,X_p)=\sum_i\beta_i\prod_{p|f_{B_{i,1}}}\tilde{F}_p(B_{i,1},X_p).$$

On the other hand, the (N,R)-th Fourier-coefficient of ϕ_1 and of $\phi_1|\Gamma^J_n\widetilde{k_{q,s}}\Gamma^J_n$ are given by :

$$\begin{aligned} A(\phi_1,(N,R))&=c(\delta_{B_1})f_{B_1}^{k-1/2}\prod_{p|f_{B_1}}\tilde{F}_p(B_1,\alpha_p),\\ A(\phi_1,(N,R),\widetilde{k_{q,s}})&=c(\delta_{B_1})f_{B_1}^{k-1/2}\sum_i\beta_i\prod_{p|f_{B_{i,1}}}\tilde{F}_p(B_{i,1},\alpha_p).\end{aligned}$$

Hence if we put $X_p=\alpha_p$ in (2.3) and multiply both sides by $c(\delta_{B_1})f_{B_1}^{k-1/2}$, we conclude that ϕ_1 is a Hecke eigenform. Hence $\Psi^{(2n-1)}(f)$ is also a Hecke eigenform.

Next we shall show the second statement of Theorem 1. Zharkovskaya's theorem is also known for half-integral weight (cf. [5]). Let $E^{(2n-1)}_{k-1/2}$ be the Siegel modular form of weight $k-1/2$ and degree $2n-1$ which corresponds to $e^{(2n-1)}_{k,1}$. By using Zharkovskaya's theorem, for any even integer $k>2n+1$, we have the following identity :

$$L(s,E^{(2n-1)}_{k-1/2})\;=\;\prod_{i=0}^{2n-2}L(s-i,E^{(1)}_{2(k-n)}),$$

(up to Euler 2-factors,) where $E^{(1)}_{2(k-n)}$ is the Eisenstein series of weight $2(k-n)$ of degree 1. This identity implies a property of $\Phi(\Gamma^J_n k_{s,q} \Gamma^J_n, X_q)$. Because $\Phi(\Gamma^J_n k_{s,q} \Gamma^J_n, \alpha_q)$ is the Hecke eigenvalue of ϕ_1 for $\Gamma^J_n k_{s,q} \Gamma^J_n$, and because the form ϕ_1 corresponds to $\Psi^{(2n-1)}(f)$, we have the identity :

$$L(s, \Psi^{(2n-1)}(f)) = \prod_{i=0}^{2n-2} L(s-i, f),$$

up to Euler 2-factors.

2.5 The proof of Theorem 2

The proof of Theorem 2 is almost the same as the proof of Theorem 1. We deduce some properties of $\tilde{F}_p(B, X_p)$ by using Yamazaki's theorem.

Let ϕ_r be the r-th Fourier-Jacobi coefficient of $I(f)$, and let $A(\phi_r, (N, R))$ be the (N, R)-th Fourier coefficient of ϕ_r for $(N, R) \in S^+_{2n-1,r}$. Then we have

$$A(\phi_r, (N, R)) = A\,(I(f), B_r) = c(\delta_{B_r}) f_{B_r}^{k-1/2} \prod_{p \mid f_{B_r}} \tilde{F}_p(B_r, \alpha_p), \tag{2.4}$$

where $B_r = \begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^tR & r \end{pmatrix}$.

Using Yamazaki's theorem, we obtain

$$A(e^{(2n-1)}_{k,r}, (N, R)) = A(e^{(2n-1)}_{k,1}, (N, R), D_{2n-1}(r, \{p^{k-n-1/2}\}_p)), \tag{2.5}$$

where $A(e^{(2n-1)}_{k,1}, (N, R), D_{2n-1}(r, \{p^{k-n-1/2}\}_p))$ is the (N, R)-th Fourier coefficient of $e^{(2n-1)}_{k,1} | D_{2n-1}(r, \{p^{k-n-1/2}\}_p)$. On the other hand, by a direct calculations, we have

$$\begin{aligned} &A(e^{(2n-1)}_{k,1}, (N, R), D_{2n-1}(r, \{p^{k-n-1/2}\}_p)) \\ &= h_{k-n-1/2}(\delta_{B_r}) f_{B_r}^{k-n-1/2} \sum_i \gamma_i \prod_{p \mid f_{B'_{i,1}}} \tilde{F}_p(B'_{i,1}, p^{k-n-1/2}), \end{aligned} \tag{2.6}$$

where the γ_i are certain constants and the $B'_{i,1}$ are certain matrices of the form $B'_{i,1} = \begin{pmatrix} N' & \frac{1}{2}R' \\ \frac{1}{2}{}^tR' & 1 \end{pmatrix}$, and where γ_i does not depend on the choice of k.

By using Lemma 2, and identities (2.4), (2.5), (2.6), we obtain

$$\prod_{p \mid f_{B_r}} \tilde{F}_p(B_r, X_p) = \sum_i \gamma_i \prod_{p \mid f_{B'_{i,1}}} \tilde{F}_p(B'_{i,1}, X_p).$$

Hence if we put $X_p = \alpha_p$ and multiply both sides by $c(\delta_{B_r}) f_{B_r}^{k-1/2}$, we have

$$A(\phi_r, (N, R)) = A(\phi_1, (N, R), D_{2n-1}(r, \{\alpha_p\}_p)).$$

Hence we conclude Theorem 2.

References

[1] S.Boecherer : Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, *Math.Z.* **183** (1983), 21–46

[2] H.Cohen : Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann* **217** (1975), 171–185.

[3] M.Eichler and D.Zagier : Theory of Jacobi Forms, Progress in Math. **55**, Birkhäuser, Boston-Basel-Stuttgart, 1985.

[4] E.Freitag : Siegelsche Modulfunktionen. *Springer, Berlin, Heidelberg, New York* 1983.

[5] S.Hayashida : Zeta function and Zharkovskaya's thorem on half integral weights Siegel modular forms. *Acta Arith* **108** no.4 (2003) 391–399.

[6] T.Ibukiyama : On Jacobi forms and Siegel modular forms of half integral weights, *Comment. Math. Univ. St.Paul.* **41** No.2 (1992) 109–124.

[7] T.Ikeda : On the lifting of elliptic cusp forms to Siegel cusp forms of degree 2n, *Ann. of Math. (2)* **154** no.3, (2001), 641–681.

[8] W.Kohnen : Modular forms of half integral weight on $\Gamma_0(4)$, *Math, Ann.* **248** (1980), 249–266.

[9] K.Takase : On Siegel modular forms of half-integral weights and Jacobi forms. *Trans. Am. Math. Soc.* **351** No.2, (1999) 735-780.

[10] T.Yamazaki : Jacobi forms and a Maass relation for Eisenstein series. *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.* **33** (1986) 295–310.

[11] T.Yamazaki : Jacobi forms and a Maass relation for Eisenstein series II, *J.Fac.Sci.Univ.Tokyo Sect. IA, Math.* **36** (1989) 373–386.

[12] Zhuravlev.V.G : Hecke rings for a covering of the symplectic group. *Math, sbornik,* **121 (163)** no.3, (1983) 381–402.

[13] Zhuravlev.V.G : Euler expansions of theta transforms of Siegel modular forms of half-integral weight and their analytic properties, *Math, sbornik,* **123 (165)** (1984), 174–194.

[14] C.Ziegler : Jacobi forms of higher degree, *Abh. Math. Sem. Univ. Hamburg,* **59** (1989), 191–224.

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